A Modified quasiparticle Model

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By use of the quasiparticle-quasihole space as model space, a model is established in order to determine excitation energies and transition probabilities of atomic nuclei. The equations derived show a remarkable correspondence to those in Migdal's quasiparticle theory. The validity of the quasiparticle hypothesis, which is fundamental in Migdal's theory, is not a necessary condition for a weak dependency of the effective interaction. This energy dependency is also independent of the choice of the one particle basis.

1. Introduction

In two previous articles [1, 2] we established general models used in order to determine matrix elements of the Green-operator in a given model space. In the works of BRANDOW [3], MACFARLANE [4], and MCVOY and ROMO [5] similar methods are used in confirming the phenomenological shell model. All of these authors use the space of the shell model states as model space. We feel, that in order to prove phenomenological shell model one should include not only well known model spaces in which the model states are well known; for example shell model states. In this paper we use, as an example for this case, the quasi-particle-quasihole space as model space, and derive a somewhat modified form of the quasiparticle theory of MIGDAL [6]. We make use of this form in order to discuss several problems which result in the quasiparticle theory.

a) In defining the effective interaction Γ^ω, the quasiparticle hypothesis is used by Migdal in which it is assumed that the one particle Green-function has one dominant pole. We would therefore expect that the energy dependency of Γ^ω no longer may be neglected, if this hypothesis is not fulfilled.

b) In the phenomenological shell model the one particle potential is fitted onto the experimental energies and onto the nuclear density. Considering Migdal's work, the one particle potential is determined by the requirement that the one particle wave function diagonalizes the Green-function. This is a very severe requirement, which will not in general, be fulfilled by usual potentials. The question arises to what extent the validity of the quasiparticle theory is dependent on the choice of the basis.

c) In practical calculations the number N of intermediate states, which have to be taken into account, is limited. In explaning this restriction, one generally supposes the other matrix elements of the effective interaction to be small enough in order to be neglected. However, this assumption does not seem to be well satisfied; the results generally depend quite considerably on N. It is simpler to define the effective interaction as dependent on N. One then has to examine the energy dependency of the effective interaction varying with N.

In section two, we briefly review the quasiparticle theory of Migdal and its use in calculating collective excitations and transition probabilities. In the following section we introduce the modified quasiparticle model used in determining excitation energies. The calculation of transition probabilities will be dealt with in section four. The effective interaction $\bar{\Gamma}^\omega(N)$ in the modified quasi particle model, is defined depending on the number N of intermediate states which have to be taken into account. In section five an equation will be derived which couples $\bar{\Gamma}^\omega(N_1)$ and $\bar{\Gamma}^\omega(N_2)$ to each other.

2. The quasiparticle Theory

In this section we give a short outline of the quasiparticle theory for calculating collective excitation energies and the corresponding transition probabilities.

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Excitation energies of the system are given through the poles in the variable $\omega_1 - \omega_4$ of the two particle Green function

$$K_{\lambda_1\lambda_2\lambda_3\lambda_4}(\omega_1\,\omega_2\,\omega_3\,\omega_4) = \frac{1}{(2\pi)^2}\int \mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}t_3\,\mathrm{d}t_4\; e^{i(\omega_1 t_1+\omega_2 t_2-\omega_3 t_3-\omega_4 t_4)}\,,$$
$$\langle E_0\,|\,T\{a_{\lambda_1}(t_1)\,a_{\lambda_2}(t_2)\,a^+_{\lambda_3}(t_3)\,a^+_{\lambda_4}(t_4)\,|\,E_0\rangle\;;\quad a^+_\lambda = e^{iHt}\int \mathrm{d}^3x\,\varphi^\times_\lambda(x)\,\psi^+(x)\,e^{-iHt}\,. \tag{2.1}$$

Here $|E_0\rangle$ is the exact ground state of the system, $\psi^+(x)$ and $\psi(x)$ are the creation and destruction operators at the point x. The scattering amplitude Γ defined by the relation

$$K(1\,2\,3\,4) = G(1\,3)\,G(2\,4) - G(1\,4)\,G(2\,3) + \sum_{1'2'3'4'} G(1\,1')\,G(2\,2')\,\Gamma(1'\,2'\,3'\,4')\,G(3'\,3)\,G(4'\,4)$$

has the same poles as K. $G(1.2)$ is here the one particle Green function

$$G(1\,2) \equiv G_{\lambda_1\lambda_2}(\omega_1)\,\delta(\omega_1-\omega_2) = (1/2\pi i)\int \mathrm{d}t_1\,\mathrm{d}t_2\,e^{i(\omega_1 t_1-\omega_2 t_2)}\cdot\langle E_0\,|\,T\{a_{\lambda_1}(t_1)\,a^+_{\lambda_2}(t_2)\}\,|\,E_0\rangle\,,$$

and we have combined the quantum number λ and the variable ω to one index.

$$(\lambda_1\,\omega_1) \equiv 1\,. \tag{2.4}$$

The Bethe Salpeter equation

$$\Gamma(1\,2\,3\,4) = I(1\,2\,3\,4) + \sum I(5'\,2\,3\,6')\,G(5\,5')\,G(6'\,6)\,\Gamma(1\,6\,5\,4)\,, \tag{2.5}$$

connects the scattering amplitude with the particle-hole irreducible interaction I. The operator I is weakly energy dependent in the variable $\omega_1 - \omega_4$. However, we are not able to use I as an effective interaction since firstly, I is strongly energy dependent in the other variables and secondly, we do not know the form of the one particle Green functions in (2.5). In order to renormalize (2.5) Migdal makes two assumptions

a) The one particle Green function G has one dominant pole. All other poles have smaller residues (quasiparticle hypothesis).

b) The wave functions φ_λ from Eq. (2.1) have been selected in such a way, that the one particle Green function becomes diagonal on this pole.

$$G_{\lambda_1\lambda_2}(\omega) = \left\{\frac{(1-n_{\lambda_1})\,a}{\omega-\varepsilon_{\lambda_1}+i\delta} + \frac{n_{\lambda_1}\,a}{\omega+\varepsilon_{\lambda_1}-i\delta}\right\}\delta_{\lambda_1\lambda_2} + R_{\lambda_1\lambda_2}(\omega), \tag{2.6}$$

$n_\lambda = 1$ when λ corresponds to a occupied state,
$n_\lambda = 0$ when λ corresponds to a unoccupied state.

The residue a, which is actually relying on λ_1 is regarded as a constant in the interesting domain. $R(\omega)$ is dealt with as being weakly ω-dependent. Further, we would like to use the total energy in the particle-hole channel

$$\omega = \omega_1 - \omega_4 = \omega_3 - \omega_2 = \omega_5 - \omega_6 \tag{2.7}$$

and the relative energies ε, ε' and $\tilde{\varepsilon}$

$$\varepsilon = \tfrac{1}{2}(\omega_1+\omega_4)\,,\quad \varepsilon' = \tfrac{1}{2}(\omega_3+\omega_2)\,,\quad \tilde{\varepsilon} = \tfrac{1}{2}(\omega_5+\omega_6)\,. \tag{2.8}$$

The product $G\cdot G$ from Eq. (2.5) can be split into a strongly and a weakly ω-dependent part.

$$G_{\lambda_5\lambda_5'}(\tilde{\varepsilon}+\tfrac{1}{2}\omega)\,G_{\lambda_6\lambda_6'}(\tilde{\varepsilon}-\tfrac{1}{2}\omega) = 2\pi i a^2\,\frac{n_{\lambda_6}-n_{\lambda_5}}{\omega-\varepsilon_{\lambda_5}+\varepsilon_{\lambda_6}}\,\delta(\tilde{\varepsilon})\,\delta_{\lambda_5\lambda_5'}\,\delta_{\lambda_6'\lambda_6}$$
$$+ B_{\lambda_5\lambda_5'\lambda_6\lambda_6'}(\tilde{\varepsilon},\omega) = \bar{A}_{\lambda_5\lambda_5'\lambda_6\lambda_6'}(\omega)\,\delta(\varepsilon) + B_{\lambda_5\lambda_5'\lambda_6\lambda_6'}(\tilde{\varepsilon}\,\omega).$$

Defining the effective interaction by the matrix equation

$$\Gamma^\omega(\varepsilon\,\varepsilon'\,\omega) \equiv \Gamma^\omega(\varepsilon+\tfrac{1}{2}\omega,\varepsilon'-\tfrac{1}{2}\omega,\varepsilon'+\tfrac{1}{2}\omega,\varepsilon-\tfrac{1}{2}\omega) = I(\varepsilon\,\varepsilon'\,\omega) + \int \mathrm{d}\varepsilon\, I(\tilde{\varepsilon},\varepsilon',\omega)\,B(\tilde{\varepsilon}\,\omega)\,\Gamma^\omega(\varepsilon\,\tilde{\varepsilon},\omega)\,;$$
$$I(\varepsilon,\varepsilon',\omega) \equiv I(\tilde{\varepsilon}+\tfrac{1}{2}\omega,\varepsilon'-\tfrac{1}{2}\omega,\varepsilon'+\tfrac{1}{2}\omega,\tilde{\varepsilon}-\tfrac{1}{2}\omega),$$

we obtain the renormalized Bethe-Salpeter equation

$$\Gamma(\varepsilon\,\varepsilon'\,\omega) = \Gamma^\omega(\varepsilon\,\varepsilon'\,\omega) + \Gamma^\omega(0,\varepsilon',\omega)\,\bar{A}(\omega)\,\Gamma(\varepsilon,0,\omega). \tag{2.11}$$

According to Migdal, the effective interaction Γ^{ω} has a weak ω-dependency. Γ^{ω} however, depends strongly on ε and ε' since we derive from (2.11) that Γ and Γ^{ω} have the same poles in ε and ε'. In order to obtain quasi-particle – quasi-hole states and the corresponding collective excitations, we have to determine the poles of $\Gamma(0,0,\omega)$ using the effective interaction $\Gamma^{\omega}(0,0,\omega)$.

Let us now direct our attention to the transition probability produced by the external field $\bar{F}$. If $\bar{F}$ has the form

$$\bar{F}(t) = F e^{i\omega t} = \sum_{\lambda\lambda'} T^0_{\lambda\lambda'} e^{i\omega t} a^+_{\lambda} a_{\lambda'}, \qquad (2.12)$$

the number of transitions per unit time from the ground state into the excited state $|E_s\rangle$ may be represented in the following way

$$w_{s0} = 2\pi\delta(\omega - E_s + E_0)\,|\langle E_0|F|E_s\rangle|^2 \qquad (2.13)$$
$$= 2\pi\delta(\omega - E_s + E_0)\,\Big|\sum_{\lambda\lambda'}\langle E_0|a^+_{\lambda}a_{\lambda'}|E_s\rangle\, T^0_{\lambda\lambda'}\Big|^2.$$

The total absorption probability $W(\omega)$, is given through the sum

$$W(\omega) = \sum_{s\neq 0} w_{s0}(\omega). \qquad (2.14)$$

Let us define the polarisation operator

$$P(t_1 - t_2) = i\langle E_0|T\{F(t_1)\,F(t_2)\}|E_0\rangle;$$
$$F(t) = e^{-iHt} F e^{iHt}. \qquad (2.15)$$

Using the Fourier transform

$$P(\omega) = \int dt\, P(t)\, e^{i\omega t}$$
$$= \sum_s \Big|\sum_{\lambda\lambda'}\langle E_0|a^+_{\lambda}a_{\lambda'}|E_s\rangle\, T^0_{\lambda\lambda'}\Big|^2$$
$$\cdot\left\{\frac{1}{\omega+E_s-E_0-i\delta} - \frac{1}{\omega-E_s+E_0+i\delta}\right\}$$
$$= P^{\mathrm{ret}}(\omega) + P^{\mathrm{av}}(\omega),$$

we derive the simple result

$$W(\omega) = 2\,\mathrm{Im}\,P(\omega) = 2\,\mathrm{Im}\,P^{\mathrm{av}}(\omega). \qquad (2.17)$$

We see that the residue of a pole of P or P^{av} completely determines the transition probability.

With the vertex T

$$T_{\lambda_1\lambda_4}(\varepsilon,\omega) = T^0_{\lambda_1\lambda_4} + \sum_{\lambda_3\lambda_2} T^0_{\lambda_3\lambda_2}\int d\varepsilon' \sum_{\lambda_5\lambda_6} G_{\lambda_5\lambda_2}(\varepsilon' + \tfrac{1}{2}\omega)$$
$$\cdot G_{\lambda_2\lambda_6}(\varepsilon' - \tfrac{1}{2}\omega)\,\Gamma_{\lambda_1\lambda_6\lambda_5\lambda_4}(\varepsilon\,\varepsilon',\omega)$$

or written in matrix form

$$T(\varepsilon,\omega) = T^0 \qquad (2.19)$$
$$+ \int d\varepsilon'\, T^0\, G(\varepsilon' + \tfrac{1}{2}\omega)\, G(\varepsilon' - \tfrac{1}{2}\omega)\,\Gamma(\varepsilon\,\varepsilon'\,\omega)$$

we have

$$P(\omega) = \frac{i}{2\pi}\int d\varepsilon\, T^0\, G(\varepsilon + \tfrac{1}{2}\omega)\, G(\varepsilon - \tfrac{1}{2}\omega)\, T(\varepsilon,\omega). \qquad (2.20)$$

In order to renormalize Eq. (2.19), we define the renormalized vertex T^{ω}

$$T^{\omega}(\varepsilon,\omega) = T^0 + T^0\int d\varepsilon'\, B(\varepsilon'\,\omega)\,\Gamma^{\omega}(\varepsilon\,\varepsilon'\,\omega) \qquad (2.21)$$

and obtain

$$T(\varepsilon,\omega) = T^{\omega}(\varepsilon,\omega) + T(0,\omega)\,\bar{A}(\omega)\,\Gamma^{\omega}(\varepsilon,0,\omega). \qquad (2.22)$$

For the above mentioned reasons, we are able to use (2.22) only for the calculation of $T(0,\omega)$. Therefore, we must also renormalize Eq. (2.20). We find

$$P(\omega) = T^{\omega}(0,\omega)\,\bar{A}(\omega)\,T(0,\omega)$$
$$+ (i/2\pi)\int d\varepsilon\, T^0\, B(\varepsilon,\omega)\, T^{\omega}(\varepsilon\,\omega). \qquad (2.23)$$

It immediately becomes clear that the second term on the right hand side contains no poles of P

$$P(\omega) = T^{\omega}(0,\omega)\,\bar{A}(\omega)\,T(0,\omega) \quad \text{for} \quad \omega \approx E_s - E_0. \qquad (2.24)$$

In this way the transition probabilities can be calculated using (2.22) and (2.24).

3. The Calculating of Collective Excitation in the Modified quasiparticle Model

In this section we introduce the quasiparticle model. We derive the same equations as in the previous section, the meaning of the operators being modified however. The effective interaction $\bar{\Gamma}^{\omega}$ is defined by the method introduced in the articles *A* and *B*, the model space being the quasiparticle – quasi-hole space. Differing from the Migdal's quasiparticle theory, in which the correct choice of the one particle basis and the validity of the quasiparticle hypothesis is required $\bar{\Gamma}^{\omega}$ can also be weakly energy-dependent if this hypothesis is not fulfilled. The choice of the one particle energies does not in any way influence the energy dependent part of $\bar{\Gamma}^{\omega}$.

Let us consider the matrix elements of the Green operator

$$\bar{K}_{\lambda_1\lambda_2\lambda_3\lambda_4}(W) = \langle E_0|a^+_{\lambda_2}a_{\lambda_1}[W-H]^{-1}a^+_{\lambda_3}a_{\lambda_4}|E_0\rangle, \qquad (3.1)$$

$\bar{K}$ corresponds to a special case of the two particle Green function (2.1) in accordance with the time ordering

$$t_3 = t_2 + \delta, \quad t_4 = t_1 + \delta, \quad t_3 > t_4. \qquad (3.2)$$

The one particle basis should be suitably extracted from the one particle model, as well as the distribution function

$$n_\lambda = \begin{cases} 1, & \text{if } \lambda \text{ is occupied,} \\ 0, & \text{if } \lambda \text{ is unoccupied.} \end{cases} \tag{3.3}$$

For reasons of convenience we assume λ to be enumerated according to the sequence of the one particle energies ε_λ. The matrix G^0 should be diagonal. It is defined by the matrix elements

$$\bar{G}^\circ_{\lambda_1\lambda_2\lambda_3\lambda_4}(W) = \begin{cases} \dfrac{a^2}{W-E_0-\varepsilon_{\lambda_1}+\varepsilon_{\lambda_2}}\,\delta_{\lambda_1\lambda_3}\,\delta_{\lambda_2\lambda_4} & \text{for } \lambda_1 \leqq \lambda_2 \\ \dfrac{-a^2}{W-E_0-\varepsilon_{\lambda_1}+\varepsilon_{\lambda_2}}\,\delta_{\lambda_1\lambda_3}\,\delta_{\lambda_2\lambda_4} & \text{for } \lambda_1 > \lambda_2 . \end{cases} \tag{3.4}$$

Here a^2 is a parameter, the sense of which will become clear in section five. Using the effective interaction $\bar{I}$ we are able to calculate K. We have in matrix form

$$\bar{K}(W) = \bar{G}^0(W) + \bar{G}^0(W)\,\bar{I}(W)\,\bar{K}(W). \tag{3.5}$$

Equation (3.5) is simultaneously the defining equation for $\bar{I}$. From A and B we know, that the poles of $\bar{I}$ appear on the eigenvalues of the operator H^1.

$$H^1 = H + \lim_{c\to\infty} c\,V^1;$$
$$V^1 = \sum_{\lambda_1\lambda_2} a^+_{\lambda_1}\,a_{\lambda_2}\,|\,E_0\rangle\,\langle E_0\,|\,a^+_{\lambda_2}\,a_{\lambda_1}\,. \tag{3.6}$$

The model space M is constructed by the states $a^+_{\lambda_1}a_{\lambda_2}\,|\,E_0\rangle$ we later reduce M to the quasiparticle – quasihole space. Further we allow all matrices to depend on ω instead of W

$$\omega = W - E_0\,. \tag{3.7}$$

The introduction of the "modified scattering amplitude" $\bar{\Gamma}$

$$\bar{K}(\omega) = \bar{G}^0(\omega) + \bar{G}^0(\omega)\,\bar{\Gamma}(\omega)\,\bar{G}^0(\omega) \tag{3.8}$$

permits us to form the equation

$$\bar{\Gamma}(\omega) = \bar{I}(\omega) + \bar{I}(\omega)\,\bar{G}^0(\omega)\,\bar{\Gamma}(\omega) \tag{3.9}$$

which is analogous to (2.5). It would be feasible to use Eq. (3.5) or (3.9) in contradiction to Eq. (2.5) in calculating excitation energies since firstly, the effective interaction $\bar{I}$ depends only on one energy variable and secondly, the form of $\bar{G}^0$ is well-known to us. However, in order to discover the analogous relationship to Eq. (2.11) we reduce the model space. The quasiparticle – quasihole space is a subspace of M and is constructed by the states

$$|\,n_{\lambda_1} - n_{\lambda_2}\,|\,a^+_{\lambda_1}\,a_{\lambda_2}\,|\,E_0\rangle\,. \tag{3.10}$$

Using the projection operator P

$$P_{\lambda_1\lambda_2\lambda_3\lambda_4} = |\,n_{\lambda_1} - n_{\lambda_2}\,|\,\delta_{\lambda_1\lambda_3}\,\delta_{\lambda_2\lambda_4} \tag{3.11}$$

we are able to eliminate the forbidden states. Analogous to Eq. (3.5) we define the new effective interaction $P\,\bar{\Gamma}^\omega\,P$ by the equation

$$P\,\bar{K}(\omega)\,P = P\,\bar{G}^0(\omega) + \bar{G}^0(\omega)\,P\,\bar{\Gamma}^\omega(\omega)\,P\,\bar{K}(\omega)\,P. \tag{3.12}$$

The poles ω_n of the effective interaction $P\,\bar{\Gamma}^\omega\,P$ are now connected with the operator H^1_P

$$H^1_P = H + \lim_{c\to\infty} c\,V^1_P\,;$$
$$V^1_P = \sum_{\lambda_1\lambda_2} |\,n_{\lambda_1} - n_{\lambda_2}\,|\,a^+_{\lambda_1}\,a_{\lambda_2}\,|\,E_0\rangle\,\langle E_0\,|\,a^+_{\lambda_2}\,a_{\lambda_1}\,,$$
$$H^1_P\,|\,E^1_{Pn}\rangle = E^1_{Pn}\,|\,E^1_{Pn}\rangle\,,\ \omega_n = E^1_{Pn} - E_0\,. \tag{3.13}$$

The difference $V^1 - V^1_P$ is a positive operator, as well as V^1 and V^1_P. Thus the poles of $\bar{I}$ are in a more favorable position than the poles of $\bar{\Gamma}^\omega$. The effective interaction $\bar{\Gamma}^\omega$ is then weakly energy dependent in a definite interval

$$\omega_1 > \omega > \omega_2 \tag{3.14}$$

providing the eigenstates with the related energies within the interval (3.14) have a projection into model space in the order of magnitude of one. Should a certain state be poorly approximated by a corresponding state in the model space, then $\bar{\Gamma}^\omega$ receives a pole within or near to the interval (3.14). It is possible, however, as we have demonstrated in B, to renounce the calculating of this state or its corresponding energy using a reduced model. This pole is then absent in $\bar{\Gamma}^\omega$ and we can again expect a weak energy dependency.

The projection operator P commutes with G^0 and we have, with $\bar{A}$ being previously defined in section two

$$\bar{A}(\omega) = P\,\bar{G}^0(\omega) = \bar{G}^0(\omega)\,P\,. \tag{3.15}$$

For calculation of $\bar{\Gamma}$ in the subspace we obtain, analogous to Eq. (2.11), the relationship

$$P\,\bar{\Gamma}(\omega)\,P = P\,\bar{\Gamma}^\omega(\omega)\,P + P\,\bar{\Gamma}^\omega(\omega)\,\bar{A}(\omega)\,\bar{\Gamma}(\omega)\,P. \tag{3.16}$$

It is also feasible to extend the definition of $\bar{\Gamma}^\omega$ into the space M

$$\bar{\Gamma}^\omega(\omega) = \bar{I}(\omega) + \bar{I}(\omega)\,(1-P)\,\bar{G}^0(\omega)\,\bar{\Gamma}^\omega(\omega). \tag{3.17}$$

From Eq. (3.9) we derive

$$\bar{\Gamma}^{-1}(\omega) = \bar{I}^{-1}(\omega) - \bar{G}^0(\omega) = \bar{\Gamma}^{\omega-1}(\omega) - P\,\bar{G}^0(\omega) \tag{3.18}$$

or

$$\bar{\Gamma}(\omega) = \bar{\Gamma}^{\omega}(\omega) + \bar{\Gamma}^{\omega}(\omega)\, \bar{A}(\omega)\, \bar{\Gamma}(\omega), \quad (3.19)$$

from which we again obtain Eq. (3.16) by projecting into the quasi-particle – quasi-hole space.

As shown in B the choice of the one particle energies has no influence on the energy dependency of $\bar{\Gamma}^{\omega}$, which is expected for Γ^{ω} in the quasiparticle theory. In the example of section five, we will return to this problem.

4. Transition Probabilities from the Modified quasiparticle Model

Considering Eq. (2.17) it becomes clear that it is not necessary to know the complete information contained in the polarization operator $P(\omega)$, if we want to calculate transition probabilities. It is only necessary to be familiar with the poles and residues of the advanced part $P^{\mathrm{av}}(\omega)$. In calculating within the modified quasiparticle model, we shall derive equations which are completely analogous to the Eqs. (2.21) and (2.24).

The advanced polarization operator may be presented in the following form

$$P^{\mathrm{av}}(\omega) = \langle E_0 \,|\, F[\omega + E_0 - H]^{-1} F \,|\, E_0 \rangle = T^0 K(\omega)\, T^0. \quad (4.1)$$

If we define the modified vertex $\bar{T}$ by

$$\bar{T}(\omega) = T^0 + \bar{\Gamma}(\omega)\, \bar{G}^0(\omega)\, T^0 \quad (4.2)$$

we obtain

$$P^{\mathrm{av}}(\omega) = T^0\, \bar{G}^0(\omega)\, \bar{T}(\omega)\,. \quad (4.3)$$

In analogy to the definition of T^{ω} we define $\bar{T}^{\omega}$ by

$$\bar{T}^{\omega}(\omega) = T^0 + \bar{\Gamma}^{\omega}(\omega)\,(1-P)\, \bar{G}^0(\omega)\, T^0. \quad (4.4)$$

In contrast to Eq. (2.20) we are quite able to make use of this formula in calculating $\bar{T}^{\omega}$ from the effective interaction $\bar{\Gamma}^{\omega}$. By converting Eq. (4.2) we arrive at

$$\bar{T}(\omega) = \bar{T}^{\omega}(\omega) + \bar{\Gamma}^{\omega}(\omega)\, \bar{A}(\omega)\, \bar{T}(\omega). \quad (4.5)$$

In order to renormalize Eq. (4.3) we split G^0 into $\bar{A}$ and $(1-P)\, G^0$, and make use of the Eqs. (4.4) and (4.5)

$$\begin{aligned} P^{\mathrm{av}}(\omega) &= T^0\, \bar{G}^0\, \bar{T} = T^0(1-P)\; \bar{G}^0 \bar{T}^{\omega} + T^0(1-P)\, \bar{G}^0\, \bar{\Gamma}^{\omega}\, \bar{A}\, \bar{T} + T^0\, \bar{A}\, \bar{T} \\ &= \bar{T}^{\omega}(\omega)\, \bar{A}(\omega)\, \bar{T}(\omega) \; + T^0(1-P)\; \bar{G}^0(\omega)\, \bar{T}^{\omega}(\omega)\,. \end{aligned} \quad (4.6)$$

The last term on the right hand side contains no poles ω_n of P^{av} and therefore we do not need to consider it.

$$P^{\mathrm{av}}(\omega) = \bar{T}^{\omega}(\omega)\, \bar{A}(\omega)\, \bar{T}(\omega) \quad \text{for } \omega \approx \omega_{\mathrm{n}}. \quad (4.7)$$

Thus, in order to calculate the transition probabilities in accordance with Eq. (2.17), we must determine $\bar{T}$ according to (4.5) and substitute it into formula (4.7). However, these equations are formally the same as in the quasiparticle theory.

5. The Dependence of the Effective Interaction on the Dimension of Model Space

In section three and four no presumptions were made concerning the dimension of model space, which is constructed by the states

$$|\, n_{\lambda_1} - n_{\lambda_2} \,|\, a^{+}_{\lambda_1} a_{\lambda_2} \,|\, E_0 \rangle\,.$$

Let us now assume a finite dimension. To simplify the discussion we shall re-name the model states

$$(\lambda_1, \lambda_2) \equiv i\,,$$
$$|\, n_{\lambda_1} - n_{\lambda_2} \,|\, a^{+}_{\lambda_1} a_{\lambda_2} \,|\, E_0 \rangle \equiv |\, i \rangle\,; \quad i = 1, 2, \ldots, N.$$

The projection operator P is now presented in the following way

$$P_{ij} = \begin{cases} \delta_{ij}\,, & \text{for} \quad i, j \leqq N\,, \\ 0\,, & \text{for} \quad i > N \quad \text{or} \quad j > N\,. \end{cases}$$

Equation (3.20) attains the explicit form

$$\bar{\Gamma}_{ij}(\omega) = \bar{\Gamma}^{w}_{ij}(\omega) + \sum_{l,m=1}^{N} \bar{\Gamma}^{w}_{il}(\omega)\, \bar{A}_{lm}(\omega)\, \bar{\Gamma}_{mj}(\omega). \quad (5.3)$$

N naturally influences the energy dependency of $\bar{\Gamma}^{\omega}$. This is exemplified by the fact that H^1 depends on N

$$H^1 = H + \lim_{c \to \infty} c \sum_{i=1}^{N} |\, i \rangle \langle\, i \,|\,. \quad (5.4)$$

Since each of the summands in (5.4) is positive definite, the low lying poles of $\bar{\Gamma}^{\omega}(N, \omega)$ move toward the value $\omega = 0$ if we decrease N. We shall now investigate the connection between $\bar{\Gamma}^{\omega}(N_1)$ and $\bar{\Gamma}^{\omega}(N_2)$. Additionally, we briefly discuss the case

$$N_1 = N_2 + 1\,. \quad (5.5)$$

From now on, we shall assume

$$N_2 < N_1, \quad P_1 P_2 = P_2, \tag{5.6}$$

P_1 and P_2 here represent the model spaces with

$$N = N_1 \quad \text{and} \quad N = N_2. \tag{5.7}$$

Let us define $\bar{\Gamma}^{\omega}(N_1)$ and $\bar{\Gamma}^{\omega}(N_2)$ according to Eq. (3.20)

$$\bar{\Gamma}(\omega) = \bar{\Gamma}^{\omega}(N_1, \omega) + \bar{\Gamma}^{\omega}(N_1, \omega)\, P_1 \bar{G}^0(\omega)\, \bar{\Gamma}(\omega) = \bar{\Gamma}^{\omega}(N_2, \omega) + \bar{\Gamma}^{\omega}(N_2, \omega)\, P_2 \bar{G}^0(\omega)\, \bar{\Gamma}(\omega). \tag{5.8}$$

Inverting this relation

$$\bar{\Gamma}^{-1}(\omega) = \bar{\Gamma}^{\omega-1}(N_1, \omega) - P_1 \bar{G}^0(\omega) = \bar{\Gamma}^{\omega-1}(N_2, \omega) - P_2 \bar{G}^0(\omega) \tag{5.9}$$

we find the desired relation between $\bar{\Gamma}^{\omega}(N_1)$ and $\bar{\Gamma}^{\omega}(N_2)$

$$\bar{\Gamma}^{\omega}(N_2\, \omega) = \bar{\Gamma}^{\omega}(N_1\, \omega) + \bar{\Gamma}^{\omega}(N_1\, \omega)\, (P_1 - P_2)\, \bar{G}^0(\omega)\, \bar{\Gamma}^{\omega}(N_2\, \omega)\,. \tag{5.10}$$

If the difference between N_1 and N_2 is one,

$$N_1 - N_2 = 1, \quad (P_1 - P_2)_{kj} = \delta_{kk_0}\, \delta_{jk_0}, \quad A_{k_0 k_0} = \frac{a^2}{\omega - \varepsilon_{\lambda_1 0} + \varepsilon_{\lambda_2 0}} = \frac{a^2}{\omega - \omega_0},$$

Equation (5.10) immediately becomes solveable and we obtain

$$\bar{\Gamma}^{w}_{kj}\ (N_2\, \omega) = \bar{\Gamma}^{w}_{kj}\ (N_1\, \omega) + \frac{\bar{\Gamma}^{w}_{kk_0}\ (N_1\, \omega)\ \bar{\Gamma}^{w}_{k_0 j}\ (N_1\, \omega)\ a^2}{\omega - \omega_0 - a^2\, \bar{\Gamma}^{w}_{k_0 k_0}\ (N_1\, \omega)}\,. \tag{5.12}$$

$\bar{\Gamma}^{\omega}(N_2)$ thus has poles at the roots of

$$\omega - \omega_0 - a^2\, \bar{\Gamma}^{w}_{k_0 k_0}\ (N_1\, \omega)\,. \tag{5.13}$$

Thus we conclude that the energy dependency of $\bar{\Gamma}^{\omega}(N_2)$ is strongly influenced by the choice of the one particle energies. On the other hand, we have shown in B that the position of the poles of an effective interaction depend only on the choice of the model space. In order to eliminate this contradiction, we examine the matrix element $\bar{\Gamma}^{w}_{k_0 k_0}\ (N_1\, \omega)$ more closely. From Eq. (3.11) we obtain

$$P_1\, \bar{\Gamma}^{\omega}(N_1, \omega) = [P_1\, \bar{G}^0(\omega)\,]^{-1} - [P_1\, \bar{K}(\omega)\, P_1]^{-1}\,. \tag{5.14}$$

Here, the inverse matrices are formed in the subspace corresponding to the projection operator P_1. The matrix element which concerns us is now given by

$$\bar{\Gamma}^{w}_{k_0 k_0}\ (N_1\, \omega) = \frac{1}{a^2}\ (\omega - \omega_0)\ - [\,(P_1\, \bar{K}(\omega)\, P_1)^{-1}]_{k_0 k_0}\,. \tag{5.15}$$

By substituting into (5.12) we finally arrive at

$$\bar{\Gamma}^{w}_{kj}\ (N_2\, \omega) = \bar{\Gamma}^{w}_{kj}\ (N_1\, \omega) + \frac{\bar{\Gamma}^{w}_{kk_0}\ (N_1, \omega)\, \bar{\Gamma}^{w}_{k_0 j}\ (N_1, \omega)}{[(P_1\, \bar{K}\, (\omega)\, P_1)^{-1}]_{k_0 k_0}} \tag{5.16}$$

Formula (5.15) also shows the significance of the parameter a^2, which influences the linear term of $\bar{\Gamma}^{\omega}$.

6. Discussion

The main concern of this article was to show the possibility of using other than the mommon model space for the foundation of the phenomenological shell model. The Eq. (3.19) or (2.11), which can be used in computing collection excitations, corresponding with those of the well-known random phase approximation. Although there are several differences between our model and that of Migdal, we believe that the conditions for the validity of the two models are the same.

A significant result of our work is, that the quasi-particle hypothesis is not a necessary condition for a weak energy dependency of $\bar{\Gamma}^{\omega}$ or Γ^{ω}. Referring to Eq. (5.14), it is not difficult to see that the position of the poles of $P\, \bar{\Gamma}^{\omega}\, P$ are independent of the one particle energies ε_{λ}. The reason for this is, that the

eigenvalues of $\bar{A}$ only have one pole each and therefore no roots appear for finite ω. If we allow more than one pole in the eigenvalues of $\bar{A}$, we are able to attain, by correct choice of the poles and residues, that the number of poles of $P\,\bar{\Gamma}^{\omega}\,P$ diminish. However, the danger lies in the fact that by an incorrect choice of the poles and residues the effective interaction $P\,\bar{\Gamma}^{\omega}\,P$ obtains additional poles.

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Representation of Symmetries and Observables in Functional Hilbert Space. II.*

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The representation of infinitesimal generators corresponding to the group representation discussed in the preceding paper is analyzed in the Hilbert space of functionals over real test functions. Explicit expressions for these unbounded operators are constructed by means of the functional derivative and by canonical operator pairs on dense domains. The behaviour under certain basis transformations is investigated, also for non-Hermitian generators. For the Hermitian ones a common, dense domain is set up where they are essentially selfadjoint. After having established a one-to-one correspondence between the real test function space and a complex Hilbert space the theory of quantum observables is applied to the functional version of a relativistic quantum field theory.

1. Introduction

In the preceding article[1] (Part I) two isomorphic realizations $\mathscr{L}^2(\mathscr{H}_r, \gamma)$ and $\mathscr{L}^2(\mathscr{H}_r, \lambda)$ of the Hilbert space of all square-integrable functionals on the real test function space $\mathscr{H}_r$ have been introduced. Their scalar products are constructed by means of the Friedrich-Shapiro integral where the integration space $\mathscr{H}_r$ is itself assumed to be a Hilbert space, and the characters γ and λ remind us of the Gaussian and Lebesgues measures. In the functional Hilbert space a group representation was investigated which comes into play in the functional formulation of quantum field theory (cf., also [2]). This kind of group representation is generated by an operator-function $\pi(U)$ (resp. by its associated variants $\bar{\pi}(U)$, $\pi_\lambda(U)$) which maps operators U of the test function space homomorphic with respect to the operator-multiplication into operators of the functional space.

In this paper we are concerned with the infinitesimal generators of the mentioned group representation. They are induced into the functional Hilbert space via a linear operator-function $\sigma(A)$ [resp. $\bar{\sigma}(A)$, $\sigma_\lambda(A)$] which maps an operator A of the test function space homomorphic with respect to the commutator-composition into a functional operator. It is shown that $\sigma(A)$ is in any case an unbounded operator whatever well-behaved A may be. Thus, all statements on $\sigma(A)$ have to be specified on which domains they are valid. Also for unbounded operators A it is possible to express $\sigma(A)$ explicitly in virtue of so called cononical operator pairs, which are analogous to the creation and annihilation operators of Fock space. In spite of the structural analogy of the functional Hilbert space and the Fock space which was also emphasized by SEGAL[3] one should not be mistaken to consider the underlying field theory as a trivial one. There are also structural deviations from Fock space

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